Angular Momentum and Quantum Indistinguishability

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Theoretical and Experimental Aspects of
the Spins-Statistics Connection and Related Symmetries.
(Trieste, Oct. 21-25, 2008)

Quantum mechanics on general configuration spaces

- Dirac (1931) $\rightarrow$ *Magnetic monopoles*.
- Bopp & Haag (1950) $\rightarrow$ *"Über die Möglichkeit von Spinmodellen"*.
- Schulman (1968) $\rightarrow$ *Path integral on* $SO(3)$.
- Laidlaw & DeWitt (1971) $\rightarrow$ *Path integral on more general configuration spaces*.
- Leinaas & Myrheim (1977) $\rightarrow$ *Fibre bundle formulation (anyons)*.
- Souriau, Konstant, ... $\rightarrow$ *Geometric Quantization*.

In general, in order to formulate a consistent quantum theory "stemming" from a classical configuration space $\mathcal{Q}$, it is necessary to consider complex vector bundles over $\mathcal{Q}$.

Different equivalence classes of bundles will give place to inequivalent quantizations of the same classical system, *i.e.*, superselection sectors.

In each case, the corresponding Hilbert space will be given by the space of square-integrable sections of the bundle, with respect to some measure.

Consider the following two spaces: S^1 and $[0, 2\pi]$. Then, from the Gelfand-Neumark Theorem, we know:

- $[0, 2\pi] \ncong S^1 \iff (C([0, 2\pi]), \|\cdot\|_\infty) \ncong (C(S^1), \|\cdot\|_\infty)$.
- $([0, 2\pi]/\sim) \cong S^1 \iff (C_p([0, 2\pi]), \|\cdot\|_\infty) \cong (C(S^1), \|\cdot\|_\infty)$.

$$\begin{array}{ccc} (C_p([0, 2\pi]), \|\cdot\|_\infty) & \xleftrightarrow{\;\cong\;} & (C(S^1), \|\cdot\|_\infty) \\ L^2\text{-compl.}\downarrow & & \downarrow L^2\text{-compl.} \\ (L^2([0, 2\pi]), dx) & & (L^2(S^1), d\theta) \end{array}$$

- Which means: $(L^2([0, 2\pi]), dx) \cong (L^2(S^1), d\theta)$, although $[0, 2\pi] \ncong S^1$ (only one separable Hilbert space!)

(A quite innocent/trivial remark!)

So, if we do not take into account the representations of either the algebra of functions or of some symmetry group, we cannot use the Hilbert spaces to distinguish the spaces.

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A related situation, of physical interest:

Example (Configuration space for two identical particles in $D = 2$)

$\mathbb{R}P^1 := S^1/\mathbb{Z}_2 \cong S^1$.

From the mathematical point of view, there is no difference between these two spaces. But from the physical point of view, there is a difference!!

B. Kuckert: Angular momentum intertwiners.
Phys. Lett. A 322, pp. 47-53 (2004).

Theorem (In two spatial dimensions)

The Spin-Statistics Connection (SSC) holds if and only if there is a unitary intertwiner U such that:

$$j_z = 2UJ_zU^\dagger$$

Remarks:

- Here, SSC means: $\kappa \stackrel{\text{def}}{=} e^{i\pi j_z} \stackrel{!}{=} e^{2\pi i s}$.
- U maps the 1-particle Hilbert space onto the 2-particle one.
- Characterization of the SSC, apparently inspired by Algebraic Quantum Field Theory.

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Theorem (In three spatial dimensions)

The Spin-statistics connection (SSC) holds if and only if there is a unitary intertwiner U such that:

$$j_z\big|_{\mathcal{H}_+} = 2UJ_zU^\dagger\big|_{\mathcal{H}_+}$$

Remark:

- $\mathcal{H}_+$: states of maximum (spin) angular momentum and positive z-parity...

Kuckert's approach is interesting because:

- It characterizes the SSC in non-relativistic Q.M. in terms of a unitary equivalence between angular momentum operators corresponding to different particle number Hilbert spaces (QFT?).
- The three dimensional part of the argument uses parity operators (CPT?).
- (I think) his approach could lead us to a physically motivated assumption we still need in order to "understand" the SSC from within (non-relativistic) Q.M.
- Relation to QFT? Causality?

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But (in my opinion) it also has a problem:

Although it is based on the idea that $\mathcal{Q} = (\mathbb{R}^d \times \cdots \times R^d \setminus \Delta)/S_n$, the use of local coordinates throughout makes a comparison with more geometric approaches difficult.

1. Does the conclusion of the theorem remain valid when reformulated in global terms?
2. Is there (in $D = 3$) some obstruction to the existence of such intertwiners?

In $D = 2$, first question is easy to answer, because:

- $S^1/\mathbb{Z}_2 \cong S^1$.
- All complex line bundles over S^1 are trivial.

- $C(S^1)$: Commutative C^*-algebra with unitary generator u and norm fixed by the condition $\|1 + e^{i\alpha u}\| = 2$.
- $C(S^1) = \mathcal{A}_+ \oplus \mathcal{A}_-$, with $\mathcal{A}_+$ generated by u^2.
- Since $\|1 + e^{i\alpha} u^2\| = 2$, setting $e_n := u^n$, we obtain an isomorphism: $\psi : \mathcal{A}_+ \to C(S^1) : e_{2n} \mapsto e_n$.
- Now define:

$$\begin{array}{rcl} \varphi : \mathcal{A}_+ & \longrightarrow & \mathbb{C} \\ a & \longmapsto & \varphi(a) := \int_{S^1} a(\theta) d\theta. \end{array}$$

$$\begin{array}{ccc} \left(\mathcal{A}_+, \|\cdot\|_\infty^{S^1}\right) & \xleftrightarrow{\;\cong\;} & \left(C(S^1), \|\cdot\|_\infty^{S^1}\right) \\ \text{GNS} \Big\downarrow \pi_\varphi & & \Big\downarrow \pi \\ \mathcal{B}\left(\left(L^2(M(\mathcal{A}_+)), d\mu_\varphi\right)\right) & & \mathcal{B}\left(\left(L^2(S^1), d\theta\right)\right) \end{array}$$

- The measure μ_φ so obtained is the one needed to construct the intertwiners.
- Global version in three dimensions? $\rightarrow$ find the differential operators corresponding to infinitesimal generators of rotations.
- Equivariant $SU(2)$ bundles: very natural from the point of view of quantization.
- In three dimensions, the situation is more involved, because of the appearance of non-trivial bundles. Approach based on quantization methods, ideas borrowed from NCG might prove useful.

Isham's approach

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(M,\mathbb{R}) & \xrightarrow{\ \jmath\ } & \mathsf{HamVF}(M) & \longrightarrow & 0. \\
 & & & & & \nwarrow{\scriptstyle P} & \uparrow{\scriptstyle \gamma} & & \\
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- M : symplectic manifold ($M = T^*\mathcal{Q}; \quad \mathcal{Q} = G/H$).
- Let $f \in C^\infty(M)$ and ξ_f the corresponding fundamental vector field. Then $\jmath(f) := -\xi_f$.
- $\mathcal{G}$: Lie group acting by symplectic transformations on M.
- $P : \mathcal{L}(\mathcal{G}) \to C^\infty(M,\mathbb{R})$ should be a Lie algebra homomorphism (obstruction to the existence of P at the level of Lie algebra cohomology!)

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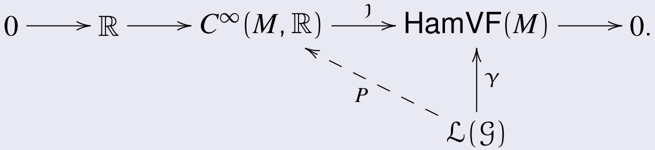

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- Look for a finite-dimensional subgroup of $C^\infty(\mathcal{Q}, \mathbb{R})/\mathbb{R} \rtimes \text{Diff}\,\mathcal{Q}$.
- Quantum observables will be given by the representations (by self-adjoint operators) of the corresponding infinitesimal generators.
- For the special case $\mathcal{Q} = G/H$, we have: $W \rtimes G$.
- In this case, the map P is naturally given by ($\tilde{A} \equiv (\varphi, A)$):

$$\begin{aligned} P : \mathcal{L}(W^* \rtimes G) &\longrightarrow C^\infty(T^*W, \mathbb{R}) \\ \tilde{A} &\longmapsto P(\tilde{A}) : (u, \psi) \mapsto \psi\left(R(A)u\right) + \varphi(u). \end{aligned}$$

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The representation space will be the space of square-integrable sections of a vector bundle E over $\mathcal{Q} = G/H$, constructed as an associated bundle to the principal bundle $G \to G/H$, by means of an irreducible unitary representation of H. For that, we need a lift of the action:

$$\begin{array}{ccc} E & \xrightarrow{l_g^{\uparrow}} & E \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{Q} & \xrightarrow{l_g} & \mathcal{Q}. \end{array}$$

Representation operators ($g \in G$):

$$(U(g)\Psi)(x) := \sqrt{\frac{d\mu_g}{d_\mu}(x)}\; l_g^{\uparrow}\Psi(g^{-1}\cdot x).$$

Results

C. Benavides & AFRL (ArXiv:0806.2449)

$\mathcal{Q} = S^2 = SU(2)/U(1)$

In this case, the obtained angular momentum operators are (locally) of the form

$$J = L - \frac{n}{2}K,$$

with n an integer. The classical expression for a charged particle in the presence of a monopole field is $\vec{J} = \vec{L} - \frac{eg}{c}\vec{K}$.

- Usually, the number n comes from compatibility conditions imposed on the wave function (winding number, Chern number, etc..)
- Here, it comes from the irrepns. of $U(1)$.

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Results

C. Benavides & AFRL (ArXiv:0806.2449)

$\mathcal{Q} = \mathbb{R}^2 = SU(2)/H$

The 2 irrepns. of

$$H := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \begin{pmatrix} 0 & \bar{\lambda} \\ -\lambda & 0 \end{pmatrix} \mid |\lambda|^2 = 1 \right\},$$

give place to fermionic/bosonic statistics.

- Bosonic case: $\overline{A}_+$, with $J_i \equiv L_i$.
- Fermionic case: $\overline{A}_-$, with $J_i \equiv L_i$.

Final remarks

- Formalism originally developed (AFRL, Ph.D. thesis-2006) in order to understand the Berry-Robbins construction.
- Applications to QPT (with H. Contreras, 2008)
- Implementability of Kuckert's approach in 3 dimensions? Interesting interplay between topology, functional analysis and physics (work in progress!)
- First step in this direction: Rotation generators for $s = 0$ particles.
- Unifying approach.

Thanks for your attention!!